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Resonant frequencies of a fluid in containers with internal bodies

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Abstract. A number of problems are solved for the resonant frequencies of oscillation of a fluid in rectangular or circular containers having internal bodies such as surface or bottom-mounted vertical blocks or circular apertures in the top surface. The variation of these frequencies with the dimensions of the bodies is obtained. The method uses matched eigenfunction expansions and Galerkin expansions to derive explicit forms for the elements S_{ij} of a 2×2 matrix required in the course of the solution. An approximate formula for an arbitrary-shaped body in a container which gives good agreement with the more accurate Galerkin approach is used to solve the resonant frequencies when the internal body is a submerged cylinder.

1. Introduction

The resonant frequencies of oscillation of fluid in a bounded container has been the subject of a number of studies over many years. Explicit solutions exist for rectangular containers in two dimensions and circular cylindrical containers in three. A good review of the existing literature is given by Moiseev and Petrov [11], whilst Fox and Kuttler [4] describe the numerical methods available for different-shaped containers. More recently Ghanimati and Naghdi [5] have considered oscillations of containers of fluid of variable bottom shape, whilst Yeung and Wu [15] have developed numerical schemes for the non-linear response of a fluid due to the impulsive motion of the container.

A different class of such sloshing problems involves the introduction of another body inside a rectangular or cylindrical container, thereby affecting the resonant frequencies. The study of these problems has application to the transportation of fluids in tanks. It is known that the effect of introducing a body into the fluid whilst keeping the same free surface, is to lower the resonant frequencies (see, for example, Moiseev [10]). In a recent paper, Evans and McIver [3] explored the effect on the frequencies of introducing a thin vertical baffle into a rectangular tank of water. The technique involved the matching of appropriate eigenfunction expansions either side of the baffle and the solution of the resulting integral equation for the horizontal fluid above or below the baffle.

In the present work we extend the technique to consider a number of such problems. Thus in Section 2 we consider the resonant frequencies of oscillation of a fluid in a rectangular container which contains either a partly immersed rectangular surface block or a totally immersed bottom-mounted rectangular block. By considering symmetric and antisymmetric potentials separately, we can construct eigenfunction expansions appropriate to two distinct fluid regions which can be matched across their common vertical boundary. The resulting integral equation can be solved using a Galerkin expansion and the required resonant frequencies can be expressed as the solution of a simple equation involving the geometry of the problem and the elements S_{ij} of a 2×2 matrix. The S_{ij} can themselves be determined to any degree of accuracy by increasing the number of terms in the Galerkin expansion.

In Section 3 we turn our attention to the axisymmetric version of this problem, namely the resonant frequencies of a fluid in a circular cylindrical container which contains a concentric partly immersed or bottom-mounted circular block. The method goes through as before with a slight modification to include the higher modes in the azimuthal direction. In Section 4 we remain with the circular container but now we consider how the resonant frequencies vary when the container is full and there is a concentric circular hole of variable radius in the otherwise closed top.

Each of the above problems reduces to solving one or more integral equations and obtaining integrated quantities S_{ij} $(i, j = 1, 2)$ from their solutions. In Section 5 the general problem is considered and it is shown how the elements S_{ij} can be determined by direct computation of certain quadratic forms. Such an approach can be shown to be equivalent to the variational approach favoured by other authors in similar problems. See, for example, Miles [8].

It was shown in [3] that a good approximation to the resonant frequencies in two-dimensional problems can be obtained using a so-called wide-spacing approximation in which it is assumed that the internal body is far from the walls of the container. Results for the baffle problem showed that even when this condition was not satisfied good accuracy was achieved over a wide range of geometrical parameters. We consider this approximation in Section 6 and use Green's theorem to derive a different but equivalent and more useful expression for the determination of the resonant frequencies, based on the assumption of wide-spacing, than that given in [3].

Results for all the problems considered are discussed in Section 7 which also includes results for a further problem, the resonant frequencies of a fluid in a rectangular container which contains a totally submerged horizontal circular cylinder. This is an approximate solution utilising the results of Section 6 and also making use of known results for the waves radiated to either infinity by the forced motion in both heave and sway of such a cylinder in a narrow wave tank. Confidence in the validity of these results for the submerged cylinder is gained by the good agreement between the results for the resonant frequencies for the problem of the submerged rectangular block using both the wide-spacing approximation and the more accurate Galerkin method.

2. Rectangular block in a rectangular container

We restrict considerations to two-dimensional motions and choose Cartesian co-ordinates x, y with y vertically downwards and $y = 0$ the undisturbed free surface. The tank occupies $|x| \leqslant b$, $0 \leqslant y \leqslant d$, and the two cases are considered together. In case I the block is bottom-mounted and occupies $|x| \leqslant a$, $a \leqslant b$, $h \leqslant y \leqslant d$, $h \leqslant d$, whilst in case II the block is assumed to be in the surface, occupying $|x| \leqslant a$, $a \leqslant b$, $0 \leqslant y \leqslant h$, $h \leqslant d$, as shown in Fig. 1.

The usual linearised theory of water waves permits the introduction of a velocity potential for the motion, which, assuming simple harmonic motions of radian frequency ω, can be expressed as

$$\Phi(x, y, t) = \operatorname{Re} \phi(x, y)\, e^{-i\omega t}. \tag{2.1}$$

Then the time-independent velocity potential $\phi(x, y)$ satisfies

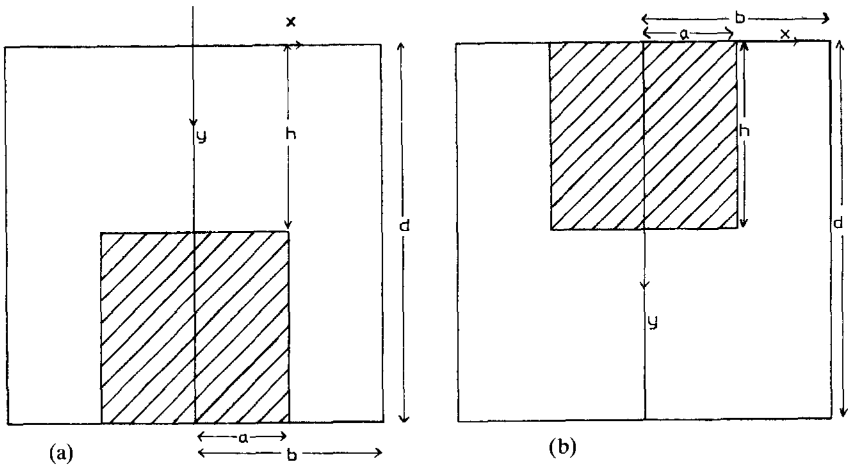


Fig. 1. Rectangular block on (a) bottom surface, (b) top surface.

$$\nabla^2\phi = 0 \quad \text{in the fluid}\,, \tag{2.2}$$

$$K\phi + \frac{\partial\phi}{\partial y} = 0\,, \quad \begin{cases} y = 0\,, & |x| < b \quad \text{(I)} \\ y = 0\,, & a < |x| < b \quad \text{(II)} \end{cases} \qquad \begin{matrix}(2.3a)\\(2.3b)\end{matrix}$$

where $K = \omega^2/g$,

$$\frac{\partial\phi}{\partial y} = 0\,, \quad \begin{cases} y = d\,, & a < |x| < b \quad \text{(I)} \\ y = d\,, & |x| < b \quad \text{(II)} \end{cases} \qquad \begin{matrix}(2.4a)\\(2.4b)\end{matrix}$$

$$\frac{\partial\phi}{\partial y} = 0\,, \quad y = h\,, \quad |x| < a \quad \text{(I), (II)} \tag{2.5}$$

$$\frac{\partial\phi}{\partial x} = 0\,, \quad x = \pm b\,, \quad 0 < y < d \quad \text{(I), (II)} \tag{2.6}$$

$$\frac{\partial\phi}{\partial x} = 0\,, \quad \begin{cases} x = \pm a\,, & h < y < d \quad \text{(I)} \\ x = \pm a\,, & 0 < y < h \quad \text{(II)}\,. \end{cases} \qquad \begin{matrix}(2.7a)\\(2.7b)\end{matrix}$$

It will be convenient to define

$$L = \{(x, y):\ x = a, 0 < y < h\} \quad \text{and} \quad L' = \{(x, y):\ x = a, h < y < d\}\,.$$

We seek functions $\phi(x, y)$ satisfying conditions (2.2) to (2.7) for particular values of $K = \omega^2/g$.

The symmetry of the problems suggests that the velocity potential be split into symmetric and anti-symmetric parts by writing

$$\phi(x, y) = \phi_s(x, y) + \phi_a(x, y)\,, \tag{2.8}$$

where

$$\phi_s(x, y) = \phi_s(-x, y)\,, \qquad \phi_a(x, y) = -\phi_a(-x, y)\,. \tag{2.9}$$

It is then clear that we can restrict attention to the region $0 < x < b$, and seek functions ϕ_s, ϕ_a each satisfying conditions (2.2) to (2.7) in that region, provided that in addition on $x = 0$, $\phi_a = 0$ and $\partial\phi_s/\partial x = 0$ for $0 < y < h$ (I) or for $h < y < d$ (II). The solution will be obtained by matching appropriate eigenfunction expansions in the two fluid regions separated by L in I and L' in II.

In the outer fluid region, $a < |x| < b$, $0 < y < d$, separation of variables gives as the most general solution for both ϕ_a, ϕ_s in both cases I and II,

$$\phi_{a,s}(x, y) = \sum_{n=0}^{\infty} B_n \Psi_n(y) \cosh k_n(b - x) , \tag{2.10}$$

where

$$\Psi_n(y) = M_n^{-1/2} \cos k_n(d - y) , \quad n = 0, 1, 2, \ldots , \tag{2.11}$$

$$K + k_n \tan k_n d = 0 , \qquad k_0 = \mathrm{i}k , \tag{2.12}$$

and

$$M_n = \frac{1}{2} (d + \sin 2k_n d/2k_n) . \tag{2.13}$$

Here B_n are unkown coefficients to be determined by the matching process. Throughout this paper it will be assumed that all unknown coefficients appearing in infinite series are such as to render those series, and series obtained by differentiation, absolutely and uniformly convergent.

It can be shown that the $\Psi_n(y)$ form a complete orthonormal set in $[0, d]$ so that

$$\int_0^d \Psi_m(y)\Psi_n(y)\,\mathrm{d}y = \delta_{mn} . \tag{2.14}$$

It follows that (2.10) satisfies (2.2), (2.3), (2.4) and (2.6).

The solution in the inner fluid region takes different forms depending on whether ϕ_a, ϕ_s or case I or II is being considered.

For case I, the bottom-mounted block, we introduce the complete orthonormal set

$$\psi_n(y) = N_n^{-1/2} \cos \kappa_n(h - y) , \quad n = 0, 1, 2 \ldots , \quad 0 < y < h , \tag{2.15}$$

where

$$K + \kappa_n \tan \kappa_n h = 0 , \qquad \kappa_0 = \mathrm{i}\kappa , \tag{2.16}$$

and

$$N_n = \frac{1}{2} (h + \sin 2\kappa_n h/2\kappa_n) , \tag{2.17}$$

whilst for case II, the surface block, we introduce the complete orthonormal set

$$\psi_n'(y) = N_n'^{-1/2} \cos \kappa_n'(h' - y') , \quad n = 0, 1, 2, \ldots , \quad 0 < y' < h' ,$$

where $h' = d - h$, $y' = y - h$, $\kappa_n' = n\pi/h'$ and

$$N_n' = h/2 , \qquad N_0' = h . \tag{2.18}$$

Separation of variables now gives, for the antisymmetric potential,

$$\phi_a(x, y) = \sum_{n=0}^{\infty} A_n \psi_n(y) \sinh \kappa_n x , \quad \text{(case I)} \tag{2.19}$$

whch satisfies (2.2), (2.3), (2.5) and also vanishes on $x = 0$ as required. The appropriate expansion in case II is obtained by replacing $\psi_n(y)$ by $\psi_n'(y)$, κ_n by κ_n' in (2.19) and $\sinh \kappa_0 x$ by x with unknown coefficients A_n' $(n = 0, 1, 2, \ldots)$.

The symmetric potential in the inner fluid region is, from separation of variables

$$\phi_s(x, y) = \begin{cases} \sum_{n=0}^{\infty} C_n \psi_n(y) \cosh \kappa_n x \quad \text{(case I)} , & (2.20) \\ \sum_{n=0}^{\infty} C_n' \psi_n'(y) \cosh \kappa_n' x \quad \text{(case II)} . & (2.21) \end{cases}$$

It remains to ensure that the potentials and velocities are continuous across L or L' and to ensure that the remaining condition (2.7) is satisfied. We focus attention on the antisymmetric potential in case I although similar arguments apply to the other potentials.

Let $U(y)$ be the horizontal fluid velocity across $x = a$. Then, from (2.10) and (2.19), continuity of horizontal velocity requires that

$$U(y) = \sum_{n=0}^{\infty} A_n \kappa_n \cosh \kappa_n a \psi_n(y) \tag{2.22}$$

$$= -\sum_{n=0}^{\infty} B_n k_n \sinh k_n c \Psi_n(y) , \quad y \in L , \tag{2.23}$$

where $c = b - a$, whence, using the orthonormality of $\{\psi_n(y)\}$, $y \in L$,

$$A_n = (\kappa_n \cosh \kappa_n a)^{-1} \langle U, \psi_n \rangle_L , \tag{2.24}$$

and, using the orthonormality of $\{\Psi_n(y)\}$, $y \in [0, d]$ and the requirement that $U(y) = 0$, $y \in L'$,

$$B_n = -(k_n \sinh k_n c)^{-1} \langle U, \Psi_n \rangle_L . \tag{2.25}$$

Here we have used the notation

$$\int_L U(y) f_n(y) \, \mathrm{d}y \equiv \langle U, f_n \rangle_L . \tag{2.26}$$

Now continuity of the potentials across $x = a$ in the fluid requires that

$$\sum_{n=0}^{\infty} A_n \psi_n(y) \sinh \kappa_n a = \sum_{n=0}^{\infty} B_n \Psi_n(y) \cosh k_n c \,, \quad y \in L \,. \tag{2.27}$$

Substitution for A_n, B_n from (2.25), (2.26) into (2.27) gives

$$\int_L U(t) G_A(y, t)\, \mathrm{d}t = k^{-1} \cot kc \langle U, \Psi_0 \rangle_L \Psi_0(y) - \kappa^{-1} \tan \kappa a \langle U, \psi_0 \rangle_L \psi_0(y) \,, \quad y \in L \,, \tag{2.28}$$

where

$$G_A(y, t) = \sum_{n=1}^{\infty} \{k_n^{-1} \coth k_n c \Psi_n(y) \Psi_n(t) + \kappa_n^{-1} \tanh \kappa_n a \psi_n(y) \psi_n(t)\} \,, \tag{2.29}$$

and we have separated out the terms corresponding to $n = 0$ so as to obtain a positive definite kernel of the integral equation derived in (2.31) below.

Let

$$U(y) = k^{-1} \cot kc \langle U, \Psi_0 \rangle_L u_2(y) - \kappa^{-1} \tan \kappa a \langle U, \psi_0 \rangle_L u_1(y) \,. \tag{2.30}$$

Then (2.28) is satisfied provided

$$\int_L u_i(t) G_A(y, t)\, \mathrm{d}t = \chi_i(y) \,, \quad y \in L \quad (i = 1, 2) \,, \tag{2.31}$$

where

$$\chi_1(y) = \psi_0(y) \,, \qquad \chi_2(y) = \Psi_0(y) \,. \tag{2.32}$$

Now, assuming $u_i(y)$ has been determined from the integral equations (2.31), we obtain, on multiplying (2.30) by Ψ_0, ψ_0 in turn and integrating over L, the two homogeneous equations

$$\begin{aligned} &\langle U, \Psi_0 \rangle (1 - k^{-1} \cot kc S_{22}) + \kappa^{-1} \tan \kappa a S_{12} \langle U, \psi_0 \rangle = 0 \,, \\ &\langle U, \psi_0 \rangle (1 + \kappa^{-1} \tan \kappa a S_{11}) - k^{-1} \cot kc S_{21} \langle U, \Psi_0 \rangle = 0 \,, \end{aligned} \tag{2.33}$$

where

$$S_{ij} = \langle u_i, \chi_j \rangle \quad (i, j = 1, 2) \,. \tag{2.34}$$

In order for these to have a non-trivial solution we require

$$\tan \kappa a \cot kc \det S - k \tan \kappa a S_{11} + \kappa \cot kc S_{22} - k\kappa = 0 \,, \tag{2.35}$$

where $\det S \equiv S_{11} S_{22} - S_{12} S_{21}$. Equation (2.35) is the key to the determination of the resonant frequencies for the anti-symmetric modes in case I, the bottom-mounted block.

For fixed dimensions of the block and tank, we can regard S_{ij} as functions of k, where

$$\omega^2/g \equiv K = k \tanh kd = -k_n \tan k_n d$$
$$= \kappa \tanh \kappa h = -\kappa_n \tan \kappa_n h \quad (n = 1, 2, \ldots)\,. \tag{2.36}$$

Thus any choice of k determines all the roots k_n, κ_n and the frequency ω, through K. The kernel $G_A(y, t)$ of the integral equations (2.31) is then determined and we can solve for u_i and hence S_{ij} from (2.34).

Having determined S_{ij} for any k, equation (2.35) can be solved for discrete values of k which in turn provide the required resonant frequencies from the relation

$$\omega^2 = gk \tanh kd\,. \tag{2.37}$$

The corresponding results for the antisymmetric modes in case II of the surface-mounted block go through in a similar manner using the appropriate dashed expressions given by (2.18). Thus the kernel of the integral equations (2.31) is modified by replacing κ_n by κ_n', whilst $x_1(y)$ becomes $\psi_0'(y)$, and the range of integration is now L' for both the integral equations and the definition of S_{ij}. Finally the equation for the determination of the resonant frequencies becomes

$$a \cot kc \det S = kaS_{11} - \cot kcS_{22} + k\,,$$

which in fact can be derived from (2.35) by letting $\kappa \to 0$.

For the symmetric modes we use (2.20) and (2.21) in the inner fluid region, the outer fluid expansion remaining the same, namely (2.10). Carrying out the same procedure as for the antisymmetric modes we find, for case I that

$$\int_L u_i(t) G_s(y, t)\,\mathrm{d}t = \chi_i(y)\,, \quad y \in L \quad (i = 1, 2)\,, \tag{2.38}$$

where now

$$G_s(y, t) = \sum_{n=1}^{\infty} \{k_n^{-1} \coth k_n c \Psi_n(y) \Psi_n(t) + \kappa_n^{-1} \coth \kappa_n a \psi_n(y) \psi_n(t)\}\,, \tag{2.39}$$

and now

$$\cot \kappa a \cot kc \det S = k \cot \kappa a S_{11} + \kappa \cot kc S_{22} - k\kappa\,, \tag{2.40}$$

where the S_{ij} are defined by (2.34) as before.

The derivation of the equation for determining the symmetric modes in case II of the surface-piercing block is slightly different although the resulting equation is the simplest of all the cases. The method proceeds as before using (2.21) in the inner fluid region down to deriving the integral equations

$$\int_{L'} u_i(t) G_s(y, t)\,\mathrm{d}t = \chi_i(y)\,, \quad y \in L'\,, \tag{2.41}$$

where

$$G_s(y,t) = \sum_{n=1}^{\infty} \{k_n^{-1} \coth k_n c \Psi_n(y)\Psi_n(t) + \kappa_n'^{-1} \coth \kappa_n' a \psi_n'(y)\psi_n'(t)\} \tag{2.42}$$

and $\chi_1(y) = \psi_0'(y)$, and where we have defined

$$U(y) = k^{-1} \cot kc \langle U, \Psi_0 \rangle_{L'} u_2(y) - C_0' u_1(y) . \tag{2.43}$$

Multiplying by Ψ_0, ψ_0' and integrating over L' now gives

$$\begin{aligned} &(1 - k^{-1} \cot kcS_{22}) \langle U, \Psi_0 \rangle_{L'} + C_0' S_{12} = 0 , \\ &k^{-1} \cot kcS_{21} \langle U, \Psi_0 \rangle_{L'} - C_0' S_{11} = 0 , \end{aligned} \tag{2.44}$$

where the condition $\langle U, \psi_0' \rangle_{L'} = 0$, arising from the requirement of mass flux conservation in this case, has been used. Equations (2.44) have a non-trivial solution if and only if

$$\cot kc \det S = kS_{11} . \tag{2.45}$$

3. Concentric circular block cylinder in a circular tank

The method of the previous section carries over with only minor modifications to the problem of determining the resonant frequencies of oscillations of a fluid in a circular cylindrical vessel containing a concentric cylindrical block either on the bottom (Case I) or piercing the surface (Case II). Cylindrical polar co-ordinates r, θ, z are chosen with origin in the undisturbed free surface, and r, z playing the rôle of x, y in Section 2. Thus the block has radius a, the tank radius is b, the depth of the tank is d and height of the block is $h - d$ (Case I) or h (Case II). We write the velocity potential $\Phi(r, \theta, z, t)$ in the form

$$\Phi(r, \theta, z, t) = \operatorname{Re} \phi_m(r, z) \, e^{\pm im\theta} \, e^{-i\omega t} , \tag{3.1}$$

where m is an integer to ensure the single-valuedness of the potential. Then $\phi_m(r, z)$ satisfies

$$\frac{\partial^2 \phi_m}{\partial r^2} + \frac{1}{r} \frac{\partial \phi_m}{\partial r} + \frac{\partial^2 \phi_m}{\partial z^2} - \frac{m^2}{r^2} \phi_m = 0 . \tag{3.2}$$

The boundary conditions on ϕ_m are the same as those satisfied by ϕ_s or ϕ_a in the last section for $0 < x < b$ given by (2.3) to (2.7), provided x, y are replaced by r, z and provided the condition at $x = 0$ is replaced by the requirement that $\phi_m(0, z)$ be bounded in the fluid.

We shall also make use of the same sets of orthonormal functions as before. Thus in the outer fluid region $a < r < b$, $0 < z < d$ separation of variables gives us the most general solution in both Cases I and II as

$$\phi_m(r, z) = \sum_{n=0}^{\infty} B_n \Psi_n(z) C_{mn}(k_n r) , \tag{3.3}$$

where

$$C_{mn}(k_n r) = I_m(k_n r)K'_m(k_n b) - I'_m(k_n b)K_m(k_n r)\,, \quad n \geq 1\,, \tag{3.4}$$

and

$$C_{m0} = J_m(kr)Y'_m(kb) - J'_m(kb)Y_m(kr)\,. \tag{3.5}$$

It is easily verified that (3.3) satisfies (3.2) and the equivalent of (2.3), (2.4) and (2.6). In the inner fluid region separation of variables gives

$$\phi_m(r, z) = A_0\psi_0(z)J_m(\kappa r) + \sum_{n=1}^{\infty} A_n\psi_n(z)I_m(\kappa_n r) \text{ (for case I)} \tag{3.6}$$

$$= A'_0\psi'_0(z)(r/a)^m + \sum_{n=1}^{\infty} A'_n\psi'_n(z)I_m(\kappa'_n r) \quad \text{(for case II)} \tag{3.7}$$

and possible solutions Y_m, K_m are excluded to ensure boundedness at $r=0$. These expressions satisfy (3.2) and the equivalent of (2.3), (2.5).

Continuity of the radial velocity requires that

$$U_m(z) = \sum_{n=0}^{\infty} B_n k_n C'_{mn}(k_n a)\Psi_n(z) \tag{3.8}$$

$$= A_0\kappa\psi_0(z)J'_m(\kappa a) + \sum_{n=1}^{\infty} \kappa_n A_n\psi_n(z)I'_m(\kappa_n a) \text{ (for case I)} \tag{3.9}$$

$$= mA'_0\psi'_0(z)a^{-1} + \sum_{n=1}^{\infty} \kappa'_n A'_n\psi'_n(z)I'_m(\kappa'_n a) \qquad \text{(for case II)} \tag{3.10}$$

whence, using the orthonormality of $\{\psi_n(z)\}$, $z \in L$, and $\psi'_n(z)$, $z \in L'$,

$$A_n = \{\kappa_n I'_m(\kappa_n a)\}^{-1}\langle U_m, \psi_n\rangle_L\,, \quad n \geq 1\,, \tag{3.11}$$

in case I, with identical relations holding involving dashed quantities in case II, whilst

$$\begin{aligned} A_0 &= \{\kappa J'_m(\kappa a)\}^{-1}\langle U_m, \psi_0\rangle_L \text{ (in case I)}\,, \\ A'_0 &= am^{-1}\langle U_m, \psi'_0\rangle_{L'} \qquad \text{(in case II)}\,. \end{aligned} \tag{3.12}$$

For $m=0$, $\langle U_0, \Psi'_0\rangle_{L'} = 0$.

The orthonormality of $\{\Psi_n(z)\rangle$, $z \in [0, d]$, and the requirement that $U(z)$ and hence $U_m(z) = 0$, $z \in L'$, yield

$$B_n = \{k_n C'_{mn}(k_n a)\}^{-1}\langle U_m, \Psi_n\rangle_L \tag{3.13}$$

for case I, with L replaced by L' in case II.

As in the two-dimensional case, we apply continuity of the potentials (3.3) and (3.6) or (3.7), substitute for A_n, B_n from (3.11) to (3.13) and separate off the terms corresponding to $n=0$ to obtain in case I:

$$\int_L u_i(t)H_m(z,t)\,\mathrm{d}t = \chi_i(z) \quad (i=1,2)\,, \quad z\in L\,, \tag{3.14}$$

$$S_{ij} = \langle u_i, \chi_j\rangle_L \quad (i,j=1,2)\,, \tag{3.15}$$

where

$$H_m(z,t) = \sum_{n=1}^{\infty}\left\{k_n^{-1}\frac{C_{mn}(k_na)\Psi_n(z)\Psi_n(t)}{C'_{mn}(k_na)} - \frac{\kappa_n^{-1}I_m(\kappa a)\psi_n(z)\psi_n(t)}{I'_m(\kappa_na)}\right\} \tag{3.16}$$

and the S_{ij} satisfy

$$\alpha\beta\det S = 1 + \beta S_{22} - \alpha S_{11}\,, \tag{3.17}$$

where

$$\alpha = J_m(\kappa a)/\kappa J'_m(\kappa a)\,, \tag{3.18}$$

$$\beta = \frac{J_m(ka)Y'_m(kb) - Y_m(ka)J'_m(kb)}{k\{J'_m(ka)Y'_m(kb) - Y'_m(ka)J'_m(kb)\}}\,. \tag{3.19}$$

For case II, κ_n, ϕ_n, L are replaced by dashed quantities defined as in (2.18) for $n>1$, whilst α in (3.18) becomes a/m. For $m=0$ equation (3.17) becomes $\beta\det S = -S_{11}$.

4. Circular cylindrical container with a concentric circular aperture

A slight modification to the previous section enables us to consider the resonant frequencies of oscillation of a liquid completely filling a circular cylindrical container of depth d and radius b which has a concentric circular opening at the top of radius $a\leqslant b$. The special case of a fluid bounded by a rigid half-plane containing a circular hole, corresponding to $b, d\to\infty$ has been considered by Miles [9].

As in Section 3 we look for a potential in cylindrical polars satisfying (3.1) and (3.2) which also satisfies

$$\frac{\partial\phi_m}{\partial z} = 0\,, \quad \begin{cases} z=d\,, & 0<r<b \quad (4.1)\\ z=0\,, & a<r<b \quad (4.2)\end{cases}$$

$$K\phi_m + \frac{\partial\phi_m}{\partial z} = 0\,, \quad z=0\,, \quad 0<r<a\,, \quad K=\omega^2/g\,, \tag{4.3}$$

$$\frac{\partial\phi_m}{\partial r} = 0\,, \quad r=b\,, \quad 0<z<d\,. \tag{4.4}$$

In the inner fluid region $0<r<a$, $0<z<d$, the expansion (3.6) is still valid, except that in the definition of $\psi_n(z)$ given by (2.15) we must replace h by d. In the outer fluid region $a<r<b$, $0<z<d$, separation of variables gives

$$\phi_m(r, z) = B_0((r/b)^m + (b/r)^m)\psi_0'(z) + \sum_{n=1}^{\infty} B_n \psi_n'(z) C_{mn}(k_n r) , \tag{4.5}$$

where C_{mn} is given by (3.4) with $k_n = n\pi/d$ and

$$\psi_0'(z) = d^{-1/2} , \qquad \psi_n'(z) = (2/d)^{1/2} \cos k_n(d - z) . \tag{4.6}$$

If we match the potential and horizontal velocities across $r = a$, $0 < z < d$ and repeat the procedure described in Section 3, we obtain equation (3.17) where α is as in (3.18),

$$\beta = \frac{a}{m} \frac{((a/b)^{2m} + 1)}{((a/b)^{2m} - 1)} , \quad (m \neq 0) \tag{4.7}$$

the integral equation is (3.14) with L extending over the full depth d, and the S_{ij} are given by (3.15) with $\chi_1 = \psi_0(z)$, $\chi_2 = \psi_0'(z)$. For $m = 0$, the equation (3.17) reduces to

$$\alpha \det S = S_{22} . \tag{4.8}$$

5. Solution of the integral equations

Common to each of the problems considered in the previous sections is the derivation of integral equations over L or L' and the need to determine integrated properties of the solutions again over L or L'.

Typically we have

$$Ku_1 = \chi_1 , \qquad Ku_2 = \chi_2 \tag{5.1}$$

where

$$Ku_1 \equiv \int_L K(z, t)u_1(t)\,\mathrm{d}t , \tag{5.2}$$

and we require

$$S_{ij} = (u_i, \chi_j) \equiv \int_L u_i(t)\chi_j(t)\,\mathrm{d}t \quad (i, j = 1, 2) , \tag{5.3}$$

where L may be replaced by L' and the kernel K may be any of G_A, G_S, H_m, or K appearing in the previous sections.

We note first that $S_{ij} = S_{ji}$, since

$$S_{12} = (u_1, \chi_2) = (u_1, Ku_2) = (Ku_1, u_2) = (u_2, Ku_1) = S_{21} . \tag{5.4}$$

We use a Galerkin approximation and expand u_i in terms of the complete set of functions ψ_n over L (or ψ_n' over L'). Thus we write

$$u_1 = \mathbf{a}^T\boldsymbol{\psi} = \sum_{n=0}^{N} a_n \psi_n , \tag{5.5}$$

$$u_2 = \mathbf{b}^T\boldsymbol{\psi} = \sum_{n=0}^{N} b_n\psi_n \,. \tag{5.6}$$

We now substitute for u_i into (5.1), multiply by ψ_m (or ψ_m') and integrate over L (or L') obtaining for $\mathbf{a}$, $\mathbf{b}$ the algebraic equations

$$\sum_{n=0}^{N} a_n(K\psi_n, \psi_m) = (\chi_1, \psi_m) \equiv \chi_{1m} \,, \text{ say} \,, \tag{5.7}$$

$$\sum_{n=0}^{N} b_n(K\psi_n, \psi_m) = (\chi_2, \psi_m) \equiv \chi_{2m} \,, \text{ say} \,. \tag{5.8}$$

In matrix notation these can be written

$$\mathbf{Ka} = \boldsymbol{\chi}_1 \,, \qquad \mathbf{Kb} = \boldsymbol{\chi}_2 \tag{5.9}$$

where

$$\mathbf{K} = \{K_{mn}\} \quad \text{and} \quad K_{mn} = (K\psi_n, \psi_m) \,. \tag{5.10}$$

Finally we have

$$\begin{aligned} S_{12} = S_{21} &= (u_1, \chi_2) = \sum_{n=0}^{N} a_n(\psi_n, \chi_2) \\ &= \mathbf{a}^T\boldsymbol{\chi}_2 = \boldsymbol{b}^T\boldsymbol{\chi}_1 \\ &= \boldsymbol{\chi}_1^T\mathbf{K}^{-1}\boldsymbol{\chi}_2 \text{ after inverting (5.9)} \,. \end{aligned} \tag{5.11}$$

In general then

$$S_{ij} = \boldsymbol{\chi}_i^T\mathbf{K}^{-1}\boldsymbol{\chi}_j \quad (i, j = 1, 2) \,, \tag{5.12}$$

and there is no need to determine the constants a_i, b_i explicitly. Jones [5] shows how (5.12) is entirely equivalent to a variational approach to (5.1) and (5.3).

6. Wide-spacing approximation

In [3] a general equation was derived from which the resonant or sloshing frequencies of oscillation of water in a rectangular tank containing an arbitrary obstacle could be determined by making use of a wide-spacing approximation. This has proved a remarkably accurate tool for solving two-dimensional radiation and scattering problems in water waves when more than one body is involved. The method is based on the assumption that the bodies are sufficiently widely-spaced for the local wave-field in the vicinity of one body not to affect the others. It has been used by Ohkusu [13] to consider the problem of two half-immersed cylinders and Srokosz and Evans [14] who considered two rolling plates. In each case good agreement was obained with more accurate solutions even when the spacing was comparable to the wavelength. For this reason Evans and McIver [3] applied the method to the sloshing problem and obtained good agreement for the sloshing frequencies (including the lowest) of a fluid in a rectangular tank containing a fixed vertical baffle. More recently Evans [1] has reconsidered the technique and has developed an alternative real equation for the determination of the sloshing frequencies which requires knowledge of the far-field wave amplitude when the body in question performs two independent rigid-body oscillations. This contrasts with the equivalent apparently complex equation derived in [3] involving the complex reflection and transmission coefficients.

The derivation of the required equation is given in [1] but for completeness it is rederived here. Recent work of Linton [7] on the heave and sway of a submerged block enables solutions derived from this equation for the sloshig frequencies in the case of the submerged block to be compared to the more accurate numerical solution of Section 2.

The derivation of the equation for the resonant frequencies for any symmetric body in a fluid region bounded by rigid walls at $x = -c, b$, is as follows.

The potential near the left-hand wall, well away from the body can be written

$$\phi(x, y) \sim \{A\,\mathrm{e}^{\mathrm{i}k(x+c)} + A\,\mathrm{e}^{-\mathrm{i}k(x+c)}\}\cosh k(d-y)\,, \tag{6.1}$$

whilst near the right-hand wall,

$$\phi(x, y) \sim \{B\,\mathrm{e}^{\mathrm{i}k(x-b)} + B\,\mathrm{e}^{-\mathrm{i}k(x-b)}\}\cosh k(d-y)\,. \tag{6.2}$$

Small heave oscillations of the body about its line of symmetry will give rise to a potential $\phi_s(x, y)$ satisfying

$$\phi_s(x, y) \sim A_s\,\mathrm{e}^{\mathrm{i}k|x|}\cosh k(d-y)\,, \quad |x| \to \infty\,, \tag{6.3}$$

whilst small sway oscillations produce a potential $\phi_a(x, y)$ satisfying

$$\phi_a(x, y) = \pm A_a\,\mathrm{e}^{\pm\mathrm{i}kx}\cosh k(d-y), \quad x \to \pm\infty\,. \tag{6.4}$$

It further follows that the functions

$$\chi_{s,a} = \phi_{s,a} - \bar{\phi}_{s,a} \tag{6.5}$$

where a bar denotes complex conjugate, has zero normal velocity on the body and

$$\chi_s(x, y) \sim (A_s\,\mathrm{e}^{\pm\mathrm{i}kx} - \bar{A}_s\,\mathrm{e}^{\mp\mathrm{i}kx})\cosh k(d-y)\,, \quad x \to \pm\infty\,, \tag{6.6}$$

whilst

$$\chi_a(x, y) \sim (\pm A_a\,\mathrm{e}^{\pm\mathrm{i}kx} \mp \bar{A}_a\,\mathrm{e}^{\mp\mathrm{i}kx})\cosh k(d-y)\,, \quad x \to \pm\infty\,, \tag{6.7}$$

We now use the identity

$$\int_C \left(\phi\,\frac{\partial\chi_{a,s}}{\partial n} - \chi_{a,s}\,\frac{\partial\phi}{\partial n}\right)\mathrm{d}s = 0\,, \tag{6.8}$$

true for sufficiently smooth harmonic functions, where C is a closed contour consisting of the boundary of the fluid closed by the lines $x = -c, b$.

Because of the conditions satisfied by ϕ, $\chi_{a,s}$ on the free surface, the bottom, and the body, the only contribution from C arises from the lines $x = -c, b$. In keeping with the wide-spacing approximation we assume that $x = +\infty, -\infty$ in (6.6), (6.7) can be replaced by $x = b, -c$ respectively so that these asymptotic forms for χ_s, and (6.1), (6.2) for $\phi(x, y)$, can be used in (6.8). We obtain

$$\cos(kc + \theta_s)A + \cos(kb + \theta_s)B = 0 \tag{6.9}$$

and

$$\cos(kc + \theta_a)A - \cos(kb + \theta_a)B = 0\,, \tag{6.10}$$

whence

$$\cos(kc + \theta_s)\cos(kb + \theta_a) + \cos(kb + \theta_s)\cos(kc + \theta_a) = 0\,. \tag{6.11}$$

For a symmetrically-placed obstacle, $c = b$ and (6.11) reduces to

$$\cos(kb + \theta_s)\cos(kb + \theta_a) = 0 \tag{6.12}$$

where

$$A_{s,a} = |A_{s,a}|\exp\{\mathrm{i}\theta_{s,a}\}\,. \tag{6.13}$$

Thus knowledge of the phase $\theta_{s,a}$ of the radiated wave due to heave or sway motion of the obstacle is all that is required to determine the resonant frequency. In fact, from (6.12),

$$kb = \begin{cases} (2n+1)\dfrac{\pi}{2} - \theta_s \\ \\ (2m+1)\dfrac{\pi}{2} - \theta_a \end{cases} \quad (m, n \text{ integers})\,. \tag{6.14}$$

This result is equivalent to that derived in [3] since their equation (3.6) reduces to

$$\begin{aligned} \mathrm{e}^{-2\mathrm{i}kb} &= R + T = -\mathrm{e}^{2\mathrm{i}\theta_s} \\ &= R - T = -\mathrm{e}^{2\mathrm{i}\theta_a} \end{aligned} \tag{6.15}$$

from the Newman relations (Newman [12]), which is equivalent to (6.12) or (6.14).

The approximate equation (6.12) for the resonant frequencies separates the dimensions of the container, b, from that of the obstacle a, h, through θ_s, θ_a.

Results for θ_s, θ_a for a submerged swaying or heaving block are available in [7] and these results will be used to compare results derived from (6.12) with results based on the full theory as described in Section 2.

The wide-spacing approximation will also be used to estimate the resonant frequencies for fluid in a rectangular container containing a horizontal, completely submerged rigid circular cylinder. All that is required are the phase angles θ_s, θ_a of the radiated waves due to forced heave and sway motion of the cylinder in an infinitely wide container of the same depth. Equation (6.14) then provides the resonant wavenumbers. Information on θ_s, θ_a is provided by work of Linton [7] and is reproduced in the Appendix to a paper by Evans and Linton [2]. The advantage of this approximation is that the obstacle need not be symmetrically positioned, the expression for the resonant wavenumbers in the general non-symmetric case being given by equation (6.11).

Results based on (6.11) and (6.14) for a cylinder in arbitrary positions will be presented, although it has to be recognised that no check is available in these cases on the accuracy of the results.

7. Results

The numerical procedure in all cases is similar, so to fix ideas we shall concentrate on the case of the submerged block in a rectangular container, case I of Section 2. The eigenvalue relation which needs to be solved in equation (2.35) where the S_{ij} are obtained from (2.34) and the u_i satisfy (2.31). As shown in Section 5, use of a Galerkin approximation obviates the requirements of solving either integral or algebraic equations. Thus

$$S_{ij} = \boldsymbol{\chi}_i^T K^{-1} \boldsymbol{\chi}_j \quad (i, j = 1, 2) \tag{7.1}$$

where

$$\chi_{im} = (\chi_i, \psi_m) = \int_0^h \chi_i(y)\psi_m(y)\,\mathrm{d}y \tag{7.2}$$

with $\chi_i(y)$, $\psi_m(y)$ given by (2.32), (2.11) and (2.15), and

$$K_{mn} = (K\psi_n, \psi_m) = \int_0^h \psi_m(t)\,\mathrm{d}t \int_0^h G_A(y, t)\psi_n(y)\,\mathrm{d}y \tag{7.3}$$

for the antisymmetric modes, with $G_A(y, t)$ given by (2.29).

The procedure adopted is as follows. First the geometry is chosen, which means fixing a/b the relative horizontal extent of the block, b/d the dimensions of the container, and h/d the relative depth of submergence of the top of the block. Secondly a value of kb is chosen from which it is possible to determine all other quantities by first using the relations (2.36) to determine k_n, κ_n, κ, $\mathbf{K}$ which then enables S_{ij} to be computed as described above.

There are two infinite processes involved here. Thus it can be seen from (7.3), (2.29) that the elements of the matrix $\mathbf{K}$ are

$$K_{mn} = \sum_{s=1}^{\infty} \{k_s^{-1} \coth k_s c\, c_{ms} c_{ns} + \kappa_s^{-1} \tanh \kappa_s a d_{ms} d_{ns}\}\,, \tag{7.4}$$

where

$$c_{ms} = \int_L \psi_m(t)\Psi_s(t)\,\mathrm{d}t\,, \qquad d_{ms} = \int_L \psi_m(t)\psi_s(t)\,\mathrm{d}t \tag{7.5}$$

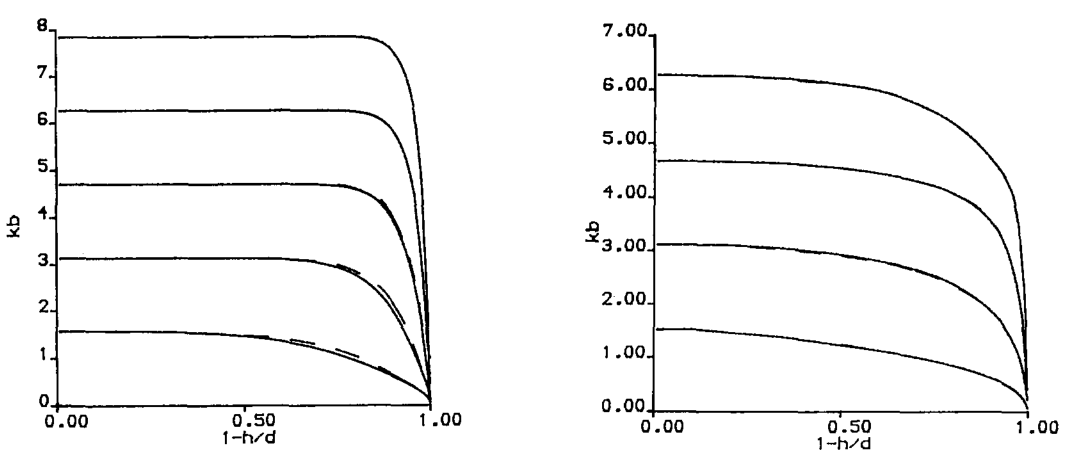
and where the explicit forms of c_{ms}, d_{ms} are $O(s^{-2})$ as $s \to \infty$. Also we need to choose a finite value of the dimension of **K** in order to obtain S_{ij} from (7.1). The infinite sum in (7.4) must also be truncated at a finite number of terms. The terms c_{ms} given by (7.5) contain the factor $\kappa_m^2 - k_s^2$ in the denominator. Now for large s, $k_s \sim s\pi/d$ so that the largest terms in the sum (7.4) occur when $s = md/h, nd/h$ and the summation must be carried out beyond $s = \max(md/h, nd/h)$ terms to obtain convergence. Again the terms d_{ms} given by (7.5) contain a factor $\kappa_m^2 - \kappa_s^2$ in the denominator. For large s, $\kappa_s \sim s\pi/h$, and the largest terms in the sum (7.5) occur when $s \simeq m, n$. Hence the terms d_{ms} converge more rapidly than the c_{ms} terms. Numerical experimentation shows that, except when the submerged block or cylinder is close to the surface, it is sufficient to choose 200 terms of the series for K_{mn} and to choose **K** to have dimension 5, in order to obtain accuracy to within two significant figures in kb.

Once we have S_{ij} all quantities in equation (2.35) can be regarded as known functions of kb. The above process is now repeated with a value of kb chosen to produce a change in sign in the left-hand side of (2.35). These two values of kb are then used in conjunction with a NAG (Numerical Algorithm Group, Oxford) library routine to determine a root kb of (2.31) lying between these original values.

The symmetric modes are determined in the same way with now the kernel replaced by (2.39) and the resonant condition by (2.40). In all other cases the procedure is similar.

With so many geometric parameters it is not feasible to give a comprehensive description of how the resonant frequencies vary with each of them. Instead we have chosen to fix the horizontal dimensions of the inner body and its container and explore how the resonant frequencies vary with an increase in the vertical extent of the inner body. Throughout we shall choose kb as the dependent variable, the actual resonant frequency being connected through $\omega^2 = gk \tanh kd$.

Figure 2 shows how kb varies with $1 - h/d$ for a rectangular block on the bottom surface for which $a/b = b/d = 0.5$. Both symmetric and antisymmetric modes are shown and it can be seen that for $h/d > 0.5$ there is little change in the resonant wavenumbers from the solution $2kb = n\pi$ $(n = 1, 2 \ldots)$ one would expect for a container of width $2b$ without an obstacle. Only when the block gets close to the surface does it begin to influence the



(a) (b)

Fig. 2. Wavenumber kb vs. $1 - h/d$ for rectangular block on bottom surface; $a/b = 0.5$, (a) $b/d = 0.5$, (b) $b/d = 2.0$. ——— Galerkin approximation, – – – Wide-spacing approximation.

resonant frequencies. The higher modes, corresponding to short wavelengths are affected even less since they penetrate only a small distance below the surface and are less sensitive to the bottom topography.

Ultimately as $h/d \to 0$, all solutions approach zero which is surprising since when $h/d = 0$, the block divides the fluid into two distinct identical regions of width $(b-a)$ and depth d and one might have expected kb to approach $m\pi/(1-a/b)$ corresponding to the solution $k(b-a) = m\pi$ $(m = 0, 1, 2, \ldots)$. It would appear therefore that the resonant wavenumber kb is discontinuous at $h/d = 0$.

Some light can be thrown upon this by considering a simpler model based on the shallow-water equations although it is recognised that these equations are not strictly appropriate for the problem since they require the depth to be small compared to a wavelength. The shallow-water equations require the solution of

$$\psi^{(1)}_{xx} + \lambda^{-1}k^2\psi^{(1)} = 0\,, \quad 0 < x < a\,, \quad \text{where } \lambda^2 = h/d\,,$$

with $\psi^{(1)}_x = 0$ on $x = 0$ for symmetric oscillations and

$$\psi^{(2)}_{xx} + k^2\psi^{(2)} = 0\,, \quad a < x < b\,,$$

with $\psi^{(2)}_x = 0$ on $x = b$, to be matched on $x = a$ so that

$$\psi^{(1)} = \psi^{(2)}\,, \qquad h\psi^{(1)}_x = \mathrm{d}\psi^{(2)}_x\,, \qquad x = a\,.$$

Thus it is a simple matter to verify that the resonant condition for symmetric oscillations using these shallow water equations is

$$\lambda \tan(ka/\lambda) + \tan k(b-a) = 0\,, \tag{7.6}$$

where here only, $k^2d = K$. Now (7.6) may be written

$$\lambda \tan(\mu X/\lambda) + \tan\{X(1-\mu)\} = 0 \tag{7.7}$$

where $\mu = a/b$ and $X = kb$ and, as $\lambda \to 0$, $\tan X(1-\mu) \to 0$ *provided* $\mu/\lambda = \mathrm{O}(1)$. Thus $k(b-a) = m\pi$ only if the ratio $a^2/hb = \mathrm{O}(1)$ as $h \to 0$. For fixed a, as $\lambda^2 = h/d \to 0$, from (7.7) we must have $\tan(\mu X/\lambda) \to \infty$ or $kb \to \lambda(n + \frac{1}{2})\pi/\mu \to 0$ as $\lambda \to 0$ for fixed μ and any given integer n. Thus we see that kb tends to zero as $h/d \to 0$, a/b fixed, and there is indeed a discontinuity in the resonant wavenumber at $h/d = 0$.

Although the above argument is based on the shallow-water approximation, a similar argument can be developed from the full linear equations by using matched asymptotic expansions, and the same conclusions found.

It is noteworthy that in the case of a thin baffle Evans and McIver [3] found that as $h/d \to 0$ the solution did indeed carry over to the solutions for two distinct regions. This is consistent with the above requirement that $a/h = \mathrm{O}(1)$ as $h/d \to 0$ since for the thin baffle $a = 0$.

Also shown in Fig. 2(a) is the solution based on the wide-spacing approximation described in Section 6. Thus kb was determined from (6.16) utilising results derived from [7] for the phase of the waves radiated by the sway and heave motion of a submerged block of the appropriate size. It can be seen that the approximation is extremely good even for $a/b = 0.5$, the only appreciable difference between the two curves occurring for the lowest resonant mode.

Figure 2(b) shows the variation of kb with $1 - h/d$ for the bottom mounted block when the horizontal dimensions are $a/b = 0.5$, $b/d = 2$, a much shallower container. Again each curve decreases monotonically to zero from its value corresponding to an empty container. The wide-spacing approximation is so good in this case that it is not possible to distinguish between the two sets of curves.

Figure 3(a) shows, for the same container, the effect of a partly immersed surface block on the resonant frequencies. Again the block occupies half the width of the container with the width of the container half its depth. As $h/d \rightarrow 0$, at the right-hand end of the curves, $kb \rightarrow 2n\pi$ $(n = 0, 1, 2, \ldots)$ corresponding to the solution $k(b - a) = n\pi$ with $a/b = 0.5$.

The first mode is antisymmetric, reducing monotonically to zero from the value of kb for the fundamental (antisymmetric) mode for an finite dock in the surface $(h/d = 1)$. The higher modes, both symmetric and antisymmetric, rapidly converge to the solution for a container of width $b - a$ reflecting the fact that these short surface waves cannot easily penetrate under the rigid surface block, even for the slightest submergence.

Figure 3(b) shows the corresponding curves for a shallower container for which $b/d = 2.0$. There is rather more variation in the curves now but again for the higher modes symmetric and antisymmetric modes rapidly tend to the limiting value $k(b - a) = n\pi$ as the depth of immersion increases.

In Fig. 4(a) the roots kb of equation (3.17) are drawn as a function of $1 - h/d$ for a submerged concentric circular block in a circular container with the ratio of both block radius to container radius and container radius to depth of one-half. The solid curves give the axisymmetric modes corresponding to no variation in the azimuthal direction $(m = 0)$ whilst the dotted curves correspond to an $\exp\{\pm i\theta\}$ variation $(m = 1)$ in the azimuthal direction. It can be seen that the influence of the block is hardly felt until it is very close to the surface, the curves differing little from the solutions of $J_0'(kb) = 0$ for $m = 0$ and $J_1'(kb) = 0$, for $m = 1$, being the conditions for the first two modes respectively in an empty

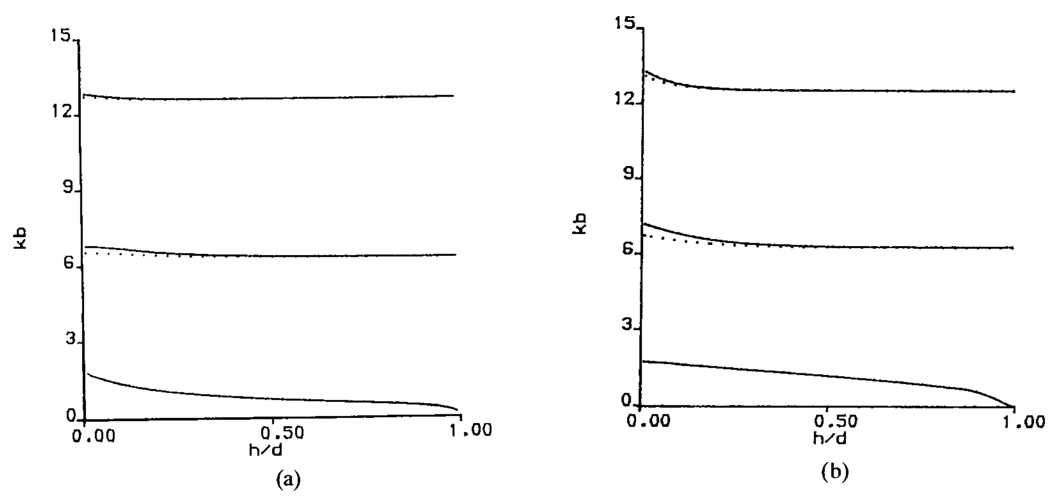


Fig. 3. Wavenumber kb vs. h/d for rectangular block on top surface; $a/b = 0.5$, (a) $b/d = 0.5$, (b) $b/d = 2.0$. ——— Anti-symmetric solution, – – – Symmetric solution.

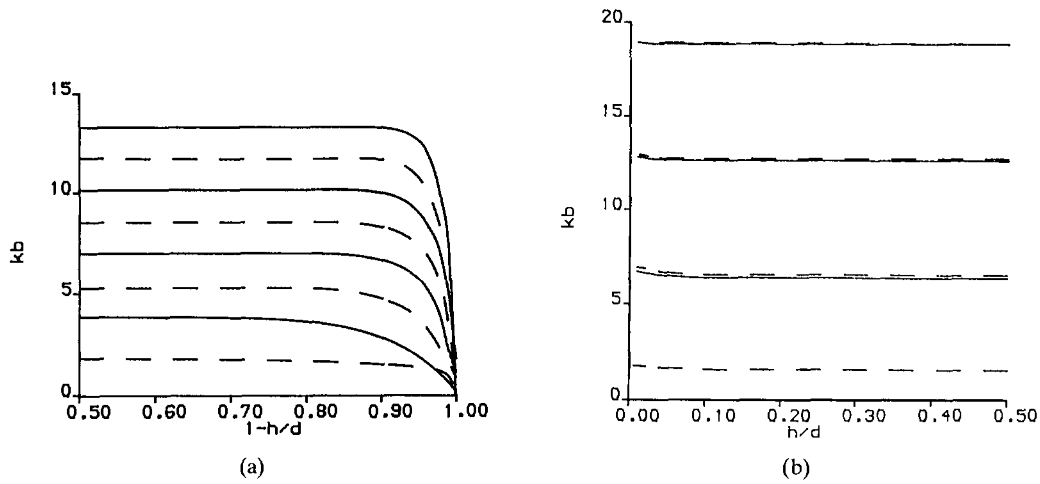


Fig. 4. Wavenumber kb vs. $1 - h/d$ for circular cylindrical block on (a) bottom surface and (b) top surface; ——— $m = 0$, - - - $m = 1$. $a/b = 0.5$, $b/d = 0.5$.

container. Again as $h/d \to 0$ the curves all approach zero and the same explanation as given for the two-dimensional block can be used here. Thus on shallow-water theory the eigenvalues turn out to be the solutions of

$$\frac{\lambda J_m'(ka/\lambda)}{J_m(ka/\lambda)} = \frac{J_m'(ka)Y_m'(kb) - J_m'(kb)Y_m'(ka)}{J_m(ka)Y_m'(kb) - J_m'(kb)Y_m(ka)} \tag{7.8}$$

with $\lambda^2 = h/d$ as before, so since the right-hand side is independent of λ, we require $J_m(ka/\lambda) \to 0$ as $\lambda \to 0$ or $kb \sim \lambda j_{mn}/\mu$ as $\lambda \to 0$, $\mu = a/b$ where j_{mn} is the nth of $J_m(z) = 0$.

Figure 4(b) shows the corresponding results for the same geometry but for a surface cylinder. The introduction of the surface block results in resonant frequencies very close to the solutions for oscillations in an annulus, namely

$$J_m'(ka)Y_m'(kb) - J_m'(kb)Y_m'(ka) = 0 \quad (m = 0, 1, \ldots) \tag{7.9}$$

over almost the entire range of h/d. Only when the block is slightly submerged is there a slight variation in the resonant wavenumber kb. It can be seen that the solutions for $m = 0$ and $m = 1$ become closer at the higher modes reflecting the closeness of the higher roots of (7.9) for $m = 0$ and $m = 1$. The solution for the first mode for $m = 0$ is absent as it would violate conservation of mass.

Results for the circular aperture in a cylindrical container discussed in Section 4 are given in Fig. 5. The size of the container is fixed at $b/d = 5$ and all curves describe the variation of resonant wavenumber kb with size of aperture a/b. For the fundamental mode $m = 0$ equation (4.8) was used, whilst for $m = 1$ equation (3.17) was used with α as in (3.18) and β as in (4.7) with $m = 1$. As $a/b \to 0$ corresponding to a vanishingly small aperture the resonant wavenumbers increase indefinitely. For increasing a/b all curves decrease monotonically to the solution of $J_0'(kb) = 0$ for $m = 0$, and $J_1'(kb) = 0$ for $m = 1$ when $a/b = 1$ corresponding to oscillations in a cylindrical open container.

In Fig. 6(a) results for a rectangular container containing a rigid submerged horizontal

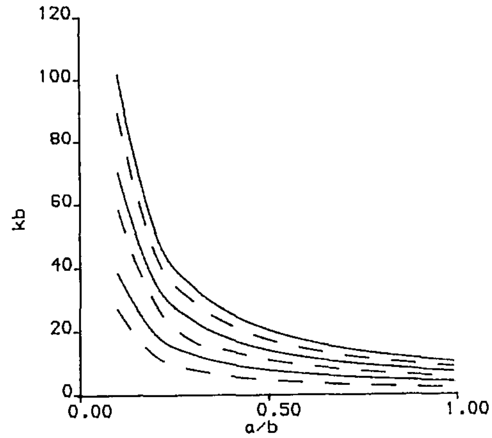
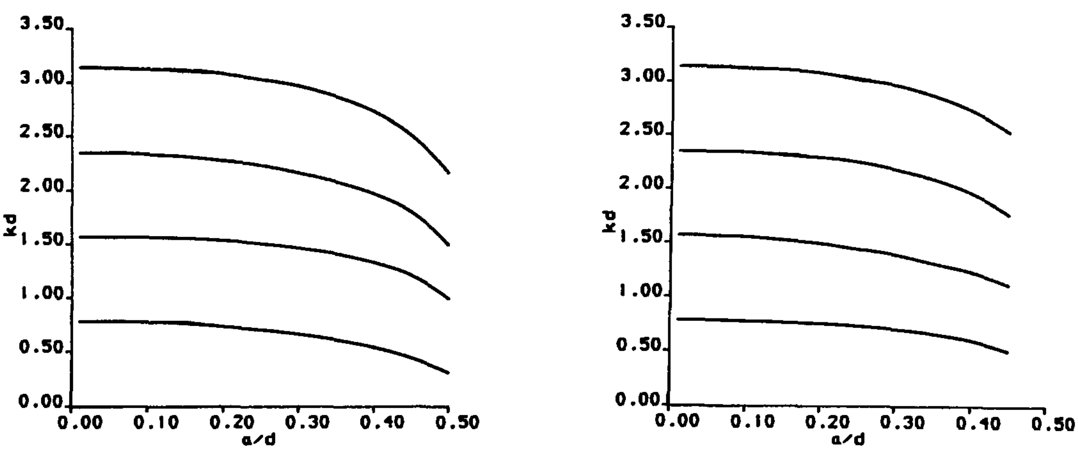


Fig. 5. Wavenumber kb vs. a/b for circular aperture on top surface; ——— $m = 0$, - - - $m = 1$. $b/d = 5.0$.

cylinder are presented using the wide-spacing approximation. The resonant wavenumber is plotted against a/d the relative size of the cylinder, for the case of a tank width to depth ratio $2b/d = 4$ and for a cylinder whose depth of centre to depth of tank ratio is $f/d = 0.5$. For $a/d \rightarrow 0$ the results approach those corresponding to a tank with no obstacle. As a/d increases the curves decrease monotonically.

In Fig. 6(b) the same geometry is used but now the cylinder is positioned to one side of the tank. Only the lower modes appear to be much affected by this shift in position, the lowest mode being increased and the second mode being decreased for all a/d up to 0.5. The solution near $a/d = 0.5$ must be treated with some caution as this corresponds to the cylinder almost filling the depth since it is more difficult to obtain accurate values of θ_s, θ_a in this situation.



(a) (b)

Fig. 6. Wavenumber kb vs. a/d for horizontal circular cylinder in a rectangular container using wide-spacing approximation; (a) $b/d = 2$, $c/d = 2$, $f/d = 0.5$, (b) $b/d = 1$, $c/d = 3$, $f/d = 0.5$.

Conclusion

A number of problems in linear water waves have been considered. The sloshing frequencies associated with the motion of a fluid with a free surface in a rectangular or circular container containing internal bodies has been determined in terms of the geometry of these bodies. In all cases the resonant frequencies reduce as the size of the bodies increase to reduce the fluid region for the same free surface in agreement with known theorems. The method is applicable to situations where eigenfunction expansions are known for separate fluid regions and is also applied to the case of a closed cylindrical full container with a concentric circular aperture at the top. An approximate solution was described and shown to give good accuracy in a particular case. This technique was then applied to the problem of determining the resonant frequencies for a rectangular container containing a submerged horizontal circular cylinder.

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